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Solving second-order ordinary differential equations by extending the Prelle–Singer method

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Abstract

We propose a method for solving second-order ordinary differential equations (ODEs) which is based on the ideas behind the Prelle–Singer (PS) procedure for first-order ODEs. While the PS procedure treats differential equations (DEs) of the form $y' = P(x, y)/Q(x, y)$, with P and Q polynomials whose coefficients lie in the field of complex numbers $\mathbb{C}$, our method is applicable to DEs of the form $y'' = P(x, y, y')/Q(x, y, y')$. The key to our approach is to focus not on the final solution but on the first-order invariants of the equation. Our method is an attempt to address algorithmically the solution of second-order ODEs with solutions in terms of elementary functions.

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1. Introduction

The fundamental role of differential equations (DEs) in scientific progress has, over the last three centuries, led to a vigorous search for methods to solve them. The overwhelming majority of these methods are based on classification of the DE into types for which a solution method is known (Kamke [1] and Murphy [2] are classic texts for this approach). This has resulted in a large variety of methods that deal with specific classes of DEs. This scene changed somewhat at the end of the 19th century when Lie [3] developed a general method to solve (or at least reduce the order of) ordinary differential equations (ODEs) given their symmetry transformations (see [4–6] for reviews of Lie's method). Lie's method is very powerful and highly general, but first requires that we find the symmetries of the DE, which may not be easy to do. Search methods have been developed [7, 8] which attempt to find symmetries of a given ODE, when these symmetries fit particular templates. However by their nature these methods are heuristic and cannot guarantee that, if an ODE has a symmetry, this will be found.

On the other hand in 1983 Prelle and Singer (PS) [9] presented a deductive method for solving first-order ODEs (FOODEs) that presents a solution in terms of elementary functions if such a solution exists [9]. (A function is elementary if it can be expressed as a combination of polynomials, logarithms and exponentials in the relevant variables. In the case of a complex field, this includes the standard trigonometric functions.) The attractiveness of the PS method lies not only in its basis on a totally different theoretical point of view but also in the fact that, if the given FOODE has a solution in terms of elementary functions, the method guarantees that this solution will be found. The method depends on the calculation of certain polynomials (called Darboux polynomials) in the elementary functions present in the ODE. Unfortunately, given a FOODE, no theoretical bound is known on the degree of the Darboux polynomials which may be required to obtain the solution. As a consequence, a terminating condition must be put in artificially by hand. In computer implementations of the PS procedure, it is usual to impose a maximum degree for the Darboux polynomials investigated, in line with the computing power available. Thus the present state of the art is that the procedure is not completely algorithmic since either an artificial cut-off has to be put in by hand, with the possible loss of solutions obtainable with a higher cut-off, or one must wait an undefined amount of time to be told that a solution does not exist. This is in contrast to the Risch integration algorithm [10] where, with a well-defined bound on the degrees of polynomials which can appear in the integral, the result of whether or not a closed-form solution exists is assured in a finite amount of time. For this reason the PS method is sometimes referred to [11] as a semi-decision procedure rather than an algorithm. If a theoretical degree bound can be found for the Darboux polynomials, the PS method will be a true algorithm.

Prelle and Singer's original paper focussed on a system of two autonomous FOODEs of the form

$$\dot{x} = P(x, y) \qquad \dot{y} = Q(x, y)$$

with P and Q polynomials in x and y with coefficients in the field of complex numbers, written $\mathbb{C}[x, y]$. Equivalently we have the form $y' = R(x, y)$, with $R(x, y)$ a rational function (a quotient of two complex polynomials) in its arguments. In subsequent papers this work has been generalized to cover solutions with non-elementary functions [12–14] and the calculation of some first integrals of autonomous systems of ODEs of higher dimension (of dimension $\geqslant 3$). Man [15] describes a method for calculating first integrals of autonomous systems which are rational or quasi-rational, but says 'The generalization of this procedure to higher dimensions to find elementary first integrals is still an open problem'. To our knowledge, the question of (semi-)algorithmic methods for the solution of SOODEs has not been addressed.

In this paper we modify the techniques developed by Prelle and Singer and apply them to second-order ODEs (SOODEs) with the rational form

$$y'' = \frac{P(x, y, y')}{Q(x, y, y')} \qquad P, Q \in \mathbb{C}[x, y, y']. \tag{1}$$

We restrict ourselves for the time being to SOODEs which have elementary solutions, i.e. which can be written in the form

$$f(x, y) = 0$$

where f is an arbitrary combination of exponentials, logarithms and polynomials in its arguments. Since we are working over a complex field, this includes standard trigonometric functions. Our goal is to find elementary first integrals of (1) when such elementary first integrals exist. We believe, given the conditions above, that these first integrals have a very particular form, described later, which permits us to construct a semi-decision procedure analogous to the PS method to find them. Once such a first integral is found, if y' can be isolated, then the PS method (or any other solution method for FOODEs) can then be applied to obtain the full solution.

2. The Prelle–Singer procedure

Despite its effectiveness in solving FOODEs, the PS procedure is not very well known outside mathematical circles. This is probably due to its non-standard approach, coupled with the fact that a computer is almost essential to realize its full efficiency. Hence we present a brief overview of the main ideas of the procedure.

Consider the autonomous system of ODEs

$$\dot{x} = Q(x, y) \qquad \dot{y} = P(x, y) \qquad P, Q \in \mathbb{C}[x, y]$$

where an overdot represents a derivative with respect to the independent variable t. This system is equivalent to the class of FOODEs which can be written as

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{P(x, y)}{Q(x, y)} \tag{2}$$

in other words those FOODEs which can be isolated in y', leaving a rational function of x and y on the right-hand side.

Prelle and Singer showed that, if an elementary first integral of (2) exists, there exists an integrating factor R with $R^n \in \mathbb{C}[x, y]$ for some integer n, such that

$$\frac{\partial RQ}{\partial x} + \frac{\partial RP}{\partial y} = 0. \tag{3}$$

The key to the success of the PS procedure is that, given the particular form of the FOODE, we know the most general form that the integrating factor can take. We can then realize a computer-assisted exhaustive search for the correct integrating factor. With the integrating factor determined, the ODE can be solved by quadrature. From (3) we see that

$$Q\,\frac{\partial R}{\partial x} + R\,\frac{\partial Q}{\partial x} + P\,\frac{\partial R}{\partial y} + R\,\frac{\partial P}{\partial y} = 0. \tag{4}$$

Thus, defining the differential operator

$$D \equiv Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y} \tag{5}$$

we have that

$$\frac{D[R]}{R} = -\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right). \tag{6}$$

Now let $R = \prod_i f_i^{n_i}$ where f_i are monic irreducible polynomials and n_i are non-zero rational numbers. From (5) we have

$$\frac{D[R]}{R} = \frac{D[\prod_i f_i^{n_i}]}{\prod_k f_k^{n_k}} = \frac{\sum_i f_i^{n_i-1} n_i D[f_i] \prod_{j\neq i} f_j^{n_j}}{\prod_k f_k^{n_k}}$$

$$= \sum_i \frac{f_i^{n_i-1} n_i D[f_i]}{f_i^{n_i}} = \sum_i \frac{n_i D[f_i]}{f_i}. \tag{7}$$

From (6), plus the fact that P and Q are polynomials, we conclude that $D[R]/R$ is a polynomial. Therefore, from (7), we see that $f_i | D[f_i]$. Written in the form

$$D[f_i] = f_i g_i \tag{8}$$

for some polynomial g_i, we see that the equation for the f_i has aspects similar to an eigenvalue equation, and for that reason f_i are sometimes called eigenpolynomials [16]. However current usage seems to prefer the term *Darboux polynomials*, and we shall refer to the f_i as such in this paper. Given an upper bound, B, on the degree of the Darboux polynomials, f_i, we thus have a criterion for finding them. We can, for example, construct all possible polynomials of degree up to B with monic leading term and arbitrary complex coefficients, construct equation (8) and see if there are non-trivial solutions for the arbitrary coefficients. With this in mind the PS procedure works as follows:

(1) Set the current degree bound, $N = 1$.
(2) Find all Darboux polynomials f_i such that $\deg f_i \leqslant N$ and $f_i|D[f_i]$.
(3) Let $D[f_i] = f_i\, g_i$. If there exist constants n_i, not all zero, such that

$$\sum_{i=1}^{m} n_i g_i = 0 \tag{9}$$

then from (3) $D[R]/R = 0$ and the ODE is exact. The solution is $w = c$, where c is an arbitrary constant and $w = \prod_{i=1}^{m} f_i^{n_i}$. If (9) has no solution then

(4) if there exist constants n_i, not all zero, such that

$$\sum_{i=1}^{m} n_i g_i = -\left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right) \tag{10}$$

then return the solution $w = c$, where c is an arbitrary constant and w is either of

$$\int RP\,\mathrm{d}x - \int \left(RQ + \frac{\mathrm{d}}{\mathrm{d}y}\int RP\,\mathrm{d}x\right)\mathrm{d}y$$

or

$$-\int RQ\,\mathrm{d}y + \int \left(RP + \frac{\mathrm{d}}{\mathrm{d}x}\int RQ\,\mathrm{d}y\right)\mathrm{d}x.$$

(5) Set $N = N + 1$. If $N > B$ then exit with no result. Else go to 2.

3. Building on the Prelle–Singer procedure

In the previous section, the main concepts used in the PS procedure were introduced. Here we present an extension of these ideas applicable to SOODEs. The basic idea is to focus on the first-order invariants of the ODE rather than on the solutions.

We shall deal with the class of SOODEs which can be written in the form

$$y'' = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{P}{Q} \qquad P, Q \in \mathbb{C}[x, y, y']. \tag{11}$$

Let us *suppose* that the ODE has a first-order invariant of the form

$$I(x, y, y') = C \tag{12}$$

with C constant on the solutions. Thus, on the solutions,

$$\mathrm{d}I = I_x\,\mathrm{d}x + I_y\,\mathrm{d}y + I_{y'}\,\mathrm{d}y' = 0 \tag{13}$$

where $I_u \equiv \frac{\partial I}{\partial u}$. Thus, on the solutions,

$$y'' = -\frac{I_x + I_y y'}{I_{y'}} \tag{14}$$

which is just (11) in terms of the invariant I. Rewriting (11) as

$$\frac{P}{Q}\,\mathrm{d}x - \mathrm{d}y' = 0 \tag{15}$$

(on the solutions) and observing that

$$y'\,\mathrm{d}x = \mathrm{d}y \tag{16}$$

we can add the identically null term $S(x, y, y')y'\,\mathrm{d}x - S(x, y, y')\,\mathrm{d}y$ to (15) and obtain that, on the solutions, the 1-form

$$\left(\frac{P}{Q} + Sy'\right)\mathrm{d}x - S\,\mathrm{d}y - \mathrm{d}y' \tag{17}$$

is identically zero. Hence, on the solutions, the 1-forms (13) and (17) must be proportional. Multiplying (17) by the factor $R(x, y, y')$, which acts as the integrating factor for (17), we have that, on the solutions,

$$\mathrm{d}I = R(\phi + Sy')\,\mathrm{d}x - RS\,\mathrm{d}y - R\,\mathrm{d}y' = 0 \tag{18}$$

where $\phi \equiv P/Q$. Comparing equations (13) and (18) we have, on the solutions, the equations

$$\begin{aligned} I_x &= R(\phi + Sy') \\ I_y &= -RS \\ I_{y'} &= -R. \end{aligned} \tag{19}$$

Now equations (19) must satisfy the compatibility conditions $I_{xy} = I_{yx}, I_{xy'} = I_{y'x}$ and $I_{yy'} = I_{y'y}$. Defining the differential operator

$$D \equiv \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial y'} \tag{20}$$

after a little algebra the compatibility conditions can be shown to be equivalent to the equations

$$D[S] = -\phi_y + S\phi_{y'} + S^2 \tag{21}$$

$$D[R] = -R(S + \phi_{y'}) \tag{22}$$

$$R_y = R_{y'}S + S_{y'}R. \tag{23}$$

Combining (21) and (22) we obtain

$$D[RS] = -R\phi_y. \tag{24}$$

We now *conjecture* that, if the ODE has an elementary solution, it is possible to choose R and S such that RS is a rational function of x, y and y'. If this is the case, by the definition of the operator D we see that $D[RS]$ will also be a rational function. Since $\phi = P/Q$ is rational (and so, therefore, is ϕ_y), equation (24) tells us that R is rational. But if R is rational and RS is rational then S is also a rational function of x, y and y'. In summary, from (19) it follows that the *supposition* that RS is rational can be equated to the existence of a first-order invariant, I, whose derivatives in relation to x, y and y' are rational functions. Obviously, if I is a rational function it follows that its derivatives also are. However the converse is not the case (see example 4 of this paper) and so we are dealing with a wider class of solutions than those which have rational first integrals.

If our conjecture holds, then our extension of the PS method applies to all SOODEs of the form (11) which have elementary solutions. Though we have not been able to prove this conjecture, extensive trials while developing this procedure have not revealed any counterexample. As supporting evidence for the veracity of the conjecture we note that, for all SOODEs with elementary solutions we have examined, when we have found two non-rational first integrals it has always been possible to construct from these first integrals another which is rational. Even if our conjecture is false, our experience with real test cases has shown that the method is, at least, applicable to the vast majority of SOODEs of the form (11) which have elementary solutions.

Since S is supposedly rational we may write

$$S = \frac{S_n}{S_d} \qquad S_n, S_d \in \mathbb{C}[x, y, y']. \tag{25}$$

Given degree bounds on S_n and S_d, the fact that (21) does not depend on R, allows us to build a set of solutions to this equation which are then candidates to solve the system of equations (21)–(23). Given these solutions, (22) implies that

$$\frac{D[R]}{R} = -(S + \phi_{y'}) = -\frac{S_n}{S_d} - \left(\frac{P}{Q}\right)_{y'} = -\frac{S_n Q^2 + S_d(Q P_{y'} - P Q_{y'})}{S_d Q^2}. \tag{26}$$

Defining the new differential operator

$$\mathcal{D} \equiv (S_d Q^2) D \tag{27}$$

equation (26) may be written as

$$\frac{\mathcal{D}[R]}{R} = -S_n Q^2 + S_d(Q P_{y'} - P Q_{y'}). \tag{28}$$

Since, as a consequence of our conjecture, R is a rational function we may write $R = \prod_i g_i^{n_i}(x, y, y')$, where $g(x, y, y')$ can be taken to be monic irreducible polynomials in $\mathbb{C}[x, y, y']$ and n_i are (possibly negative) integers. By arguments similar to those in section 2,

$$\frac{\mathcal{D}[R]}{R} = \sum_i \frac{n_i\, \mathcal{D}[g_i]}{g_i} = -S_n Q^2 + S_d(Q P_{y'} - P Q_{y'})$$

and so the polynomial factors, g_i, of the numerator and denominator of R must divide $\mathcal{D}[g_i]$. Given a degree bound on these polynomials we can find R by constructing all $g_i(x, y, y')$ up to the degree bound and testing for this division property.

Once R and S have been determined using equations (19) we have all the partial first derivatives of the first-order differential invariant, $I(x, y, y')$, which is constant on the solutions. This invariant can then be obtained as

$$\begin{aligned} I(x, y, y') = &\int R\left(\phi + S y'\right) \mathrm{d}x - \int \left[RS + \frac{\partial}{\partial y} \int R\left(\phi + S y'\right) \mathrm{d}x \right] \mathrm{d}y \\ &- \int \left\{ R + \frac{\partial}{\partial y'} \left(\int R(\phi + S y')\mathrm{d}x - \int \left[RS + \frac{\partial}{\partial y} \int R(\phi + S y')\mathrm{d}x \right] \mathrm{d}y \right) \right\} \mathrm{d}y'. \end{aligned} \tag{29}$$

If the equation $I(x, y, y') = C$ can be isolated for y', an attempt to solve the reduced ODE

$$y' = \varphi(x, y, C) \tag{30}$$

may be made using the PS method for FOODEs, or whatever solution method is appropriate. As mentioned before, the original PS method fails to be what is strictly an algorithm because no theoretical degree bound is yet known for the candidate polynomials which enter in the prospective solution, and so the procedure has no effective terminating condition for the case when an elementary solution does not exist. In practice, a terminating condition is put in by hand (it is found that polynomials of degree higher than 4 lead to computations which are overly complex for a reasonable desktop computer). However, should degree bounds for R and S be established, and our conjecture relating elementary solutions of SOODEs and their first integrals be shown to be true, then the method proposed here would be an algorithm for deciding whether first integrals exist to SOODEs of the form (1).

4. Examples

In this section we present examples of SOODEs, mostly physically motivated, that are solved by our procedure[1]. As a simple illustrative example, we begin with the classical harmonic oscillator, then consider two nonlinear SOODEs which arise from astrophysics and general relativity and finally present an example where the first integral is not rational.

Example 1 (The simple harmonic oscillator). In its simplest form, the equation for the simple harmonic oscillator is

$$y'' = -y. \tag{31}$$

For this ODE equations (21)–(23) are

$$S_x + y' S_y - y S_{y'} = 1 + S^2 \tag{32}$$

$$R_x + y' R_y - y R_{y'} = -RS \tag{33}$$

$$R_y - R_{y'} S - S_{y'} R = 0. \tag{34}$$

We initially place a degree bound $|N| = 1$ on the polynomials which can appear in R and S. Focussing first on the equation for S, we quickly obtain the solution $S = y/y'$. The differential operator

$$\mathcal{D} = y' \frac{\partial}{\partial x} + y'^2 \frac{\partial}{\partial y} - yy' \frac{\partial}{\partial y'}.$$

We find that $y' | \mathcal{D}[y'] = -yy'$ and so y' is a Darboux polynomial of degree 1 for this system. Substituting $S = y/y'$ and $R = y'$ in (34) shows that these rational functions are solutions of the system. Noting that $\phi + Sy' = 0$ for this example, on substituting in (29) we obtain

$$I(x, y, y') = -\int y \, dy - \int y' \, dy' \Rightarrow y^2 + y'^2 = C$$

as the reduced ODE, representing conservation of energy for the oscillator. This example is very simple, with ϕ independent of x and y and, as with all linear ODEs, alternative and more straightforward solution methods exist. The other examples illustrate the solution method at work for nonlinear SOODEs of the form (1).

Example 2 (An exact solution in general relativity). A rich source of nonlinear DEs in physics are the highly nonlinear equations of general relativity. Einstein's equations are, of course, in general, partial DEs, but there exist classes of space-times where the symmetry imposed reduces these equations to ODEs in one independent variable. One such class is that of static, spherically symmetric space-times, which depend only on the radial variable, r. The metric for a general static, spherically symmetric space-time has two free functions, $\lambda(r)$ and $\mu(r)$ say. On imposing the condition that the matter in the space-time is a perfect fluid, Einstein's equations reduce to two coupled ODEs for $\lambda(r)$ and $\mu(r)$. Specifying one of these functions reduces the problem to solving an ODE (of first or second order) for the other.

Following this procedure, Buchdahl [17] obtained an exact solution for a relativistic fluid sphere by considering the so-called isotropic metric

$$ds^2 = (1-f)^2(1+f)^{-2} \, dt^2 - (1+f)^4[dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)]$$

with $f = f(r)$. The field equations for $f(r)$ reduce to

$$ff'' - 3f'^2 - r^{-1} ff' = 0.$$

[1] We present only the reduction of the SOODEs since the integration of the resulting FOODE can be achieved by various methods, including the PS method itself.

Changing to the notation of this paper, with $y(x) = f(r)$, equations (21)–(23) assume the form

$$\begin{aligned} &S_x + y' S_y + \frac{y'(3xy'+y)}{xy} S_{y'} = \frac{3y'^2}{y^2} + S\frac{6xy'+y}{xy} + S^2 \\ &R_x + y' R_y + \frac{y'(3xy'+y)}{xy} R_{y'} = -RS - R\frac{6xy'+y}{xy} \\ &R_y - R_{y'} S - S_{y'} R = 0. \end{aligned} \tag{35}$$

Testing for rational solutions of (21) with $|N| = 1$ we find $S = -3y'/y$, which leads to $R = 1/xy^3$. Substituting into (29) we obtain the reduced FOODE $C = y'/xy^3$.

Example 3 (A static gaseous general-relativistic fluid sphere). In a later paper [18], Buchdahl approaches the problem of the general-relativistic fluid sphere using a different coordinate system from the previous example. For ease in comparison of the originals, we substitute Buchdahl's $\xi(r)$ by $y(x)$. Einstein's field equations then lead to

$$y'' = \frac{x^2y'^2 + y^2 - 1}{x^2 y}. \tag{36}$$

For this SOODE (21)–(23) become

$$\begin{aligned} &S_x + y' S_y + \frac{x^2y'^2 + y^2 - 1}{x^2 y} S_{y'} = \frac{x^2y'^2 - y^2 - 1}{x^2 y} + 2\frac{y'}{y} S + S^2 \\ &R_x + y' R_y + \frac{x^2y'^2 + y^2 - 1}{x^2 y} R_{y'} = -RS - 2R\frac{y'}{y} \\ &R_y - R_{y'} S - S_{y'} R = 0. \end{aligned} \tag{37}$$

This time a solution is found with $N = 4$, namely

$$S = \frac{-x^2y'^2 - xyy' + 1}{xy^2 + x^2yy'} \qquad R = \frac{y + xy'}{xy^2}. \tag{38}$$

Substituting in (29), we get the reduced FOODE

$$C = \frac{2xyy' + y^2 + x^2y'^2 - 1}{2x^2y^2}. \tag{39}$$

It is perhaps worth noting that our method manages to find this first integral while more established ODE solvers which also look for first integrals (for example the solver in MapleV.5 which is the most recent to which we have access) do not.

Example 4 (The first integral is not a rational function). At this point the reader might be thinking that the method we have described is no big deal. All the examples lead to rational first integrals so we might well just have built a procedure around constructing rational first integrals of ever higher degrees with arbitrary coefficients, differentiating them and trying to reconcile the coefficients with the original ODE. However, just because y'' can be expressed as a rational function does not of course mean that the resulting first integral is also rational. Here we show that our method can also be used to find non-rational first integrals. Consider the SOODE

$$xyy'' + yy' + 2xyy'^2 + xy'^2 = 0 \tag{40}$$

which we have constructed to prove our point, rather than for any physical relevance. This ODE fits our model with $P = y'(y + 2xyy' + xy')$ and $Q = xy$. With $N = 2$ we find the solution

$$S = \frac{y'(2y+1)}{y}$$

to (21). The solution to (22) is then $R = -1/2y'$ and, on substituting these into (29), we obtain the non-rational first integral

$$I = y + \frac{1}{2}\ln(xyy')$$

from which y' can be isolated and the resulting ODE integrated, if desired.

5. Conclusion

In this paper we have presented an approach to finding first integrals of SOODEs that is an extension of the ideas developed by Prelle and Singer [9] to tackle FOODEs. We believe it to be the first technique to address algorithmically the solution of SOODEs with elementary solutions. Our approach is based on the conjecture that, for SOODEs of the form $y'' = P(x, y, y')/Q(x, y, y')$, with P and Q polynomials with coefficients in the field of complex numbers, if an elementary solution exists then there exists at least one elementary first integral, $I(x, y, y')$, whose derivatives are all rational functions of x, y and y'. Unfortunately we have not been able to prove this conjecture, which we leave as an open problem. However, tests on ODE databases have not encountered any counterexamples and applications to SOODEs of the form treated show that our method obtains first integrals which present-day ODE solvers do not find. Hence, independent of the veracity of our conjecture, we believe the method described in this paper to be a useful addition to the set of tools for solving SOODEs. We are preparing a computational package implementing the PS procedure (and our present extension) in Maple.

Though not all SOODEs can be written in the form (1), (for example an ODE may be nonlinear in y'', may include algebraic terms or, more likely, contain non-polynomial functions such as logarithms and exponentials), a large number of them can be treated with our extension to the PS method. We also believe that our technique can be extended along lines similar to Shtokhamer's extensions of the original PS method, to deal with SOODEs where $\phi(x, y, y')$ depends on elementary functions of x, y and y', and are presently working on those ideas.

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